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Critical Behavior in a Model of Correlated Percolation

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We study the critical behavior of certain two-parameter families of correlated percolation models related to the Ising model on the triangular and square lattices, respectively. These percolation models can be considered as interpolating between the percolation model given by the + and − clusters and the Fortuin–Kasteleyn correlated percolation model associated to the Ising model. We find numerically on both lattices a two-dimensional critical region in which the expected cluster size diverges, yet there is no percolation.

KEY WORDS: Percolation; Ising model.

1. GENERAL DISCUSSION OF THE PROBLEM

Consider the standard Ising model on a simple cubic lattice in D dimensions. It is well known that for sufficiently low temperature in $D \geqslant 2$ the system exhibits long-range order (l.r.o.). It is a natural question to ask whether such a phase continues to exist if one dilutes bonds randomly with probability p. Georgii[1] provided the answer: if $p < p_c$ (the critical bond density below which a percolating cluster of occupied bonds still exists), there is an inverse temperature $\beta_c(p)$ such that for $\beta < \beta_c(p)$, l.r.o. still persists. Georgii's proof, which is based on the use of convergent low-temperature expansions, suggests straightforward generalizations to some percolation problems. Thus, one could prove, for instance, that if one randomly dilutes bonds from a square lattice, then on such a lattice Bernoulli site percolation can still occur, provided the bond dilution prob-

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ability p satisfies $p < p_c$ (i.e., the lattice does not fall apart). In the previous statement, site percolation is taken to mean the following: consider the set S of sites belonging to the infinite cluster of occupied (undiluted) bonds. Randomly with probability q and $1-q$, respectively, assign to these sites the values 1 or 0. Then percolation means that for q sufficiently large any point of S has a nonzero probability to be connected to infinity via a set of adjacent 1's.

It is well known that in percolation problems an interesting quantity is the "mean cluster size" (the expected size of the cluster attached to the origin). The latter can be infinite even though there is no percolation. A trivial example for this is provided by Bernoulli site percolation at $p = p_c$. One can also envisage situations—see examples given below—where, on the original undiluted lattice, one knows that the mean cluster size is infinite, yet there is no percolation; by analogy with the question discussed in the previous paragraph, one may ask now if this property—divergence of the mean cluster size without percolation—is stable against a sufficiently weak random bond dilution.

In our studies of the phase structure of 2D $O(N)$ spin models,[2-4] which are based on representating these models as correlated percolation models by generalizing the Fortuin–Kasteleyn representation of the Ising model,[5] we encountered a more subtle stability problem. First one forms the so-called hemispherical (H) clusters, which are constructed as follows: a bond $\langle xy \rangle$ is occupied if and only if the spins $S(x)$ and $S(y)$ point in the same hemisphere, i.e., their Nth components have equal sign (this is an obvious generalization of the + and − clusters of the Ising model, considered as bond clusters). The Fortuin–Kasteleyn (FK) clusters are then obtained from the H clusters by deleting bonds with a certain probability that goes to zero as the temperature goes to zero. Since one is interested, for instance, in proving that the mean size of the FK clusters diverges at low temperature, a natural strategy is to first show the divergence of the hemispherical clusters and then show that this property is stable under small dilutions. (In ref. 3 this problem was avoided by introducing a modification of the models, called "cut" or "constrained" models there).

We conjecture that a general stability principle holds that can be stated as follows:

Conjecture 1. If in a translation-invariant percolation model the mean size of the cluster attached to the origin (in the following simply called "mean cluster size") diverges, then, except at nongeneric critical points, the system will show stability against dilution in the following sense: there is a number $p_c > 0$ such that after deleting sites or bonds from the model with probability $p' < p_c$, the mean cluster size still diverges.

In the remainder of this paper we will limit ourselves to the case of two dimensions. In this case Conjecture 1 is implied by the following stronger statement:

Conjecture 1'. Consider a translation-invariant bond percolation model parametrized by some variable p. Eliminate bonds independently with probability p', giving rise to a 2-parameter family of percolation models parametrized by (p, p'). Let $P(L, p, p')$ be the probability that a box of size L is surrounded by a circuit of occupied bonds and $P(\infty, p, p')$ its limit as $L \to \infty$. Then, except possibly for isolated values of p, these two probabilities are continuous functions of p' at $p' = 0$.

The relation between Conjectures 1 and 1' is the following: assume that we are in a pure phase, such that the σ-algebra of translation-invariant events at infinity is trivial. Then $P(\infty, p, p')$, being the probability of a tail event, will be 0 or 1. Conjecture 1' then says that if $P(\infty, p, 0) = 1$, then also $P(\infty, p, p') = 1$ for p' sufficiently small, except for isolated values of p. The property $P(\infty, p, p') = 1$ says that any box is surrounded by a circuit of occupied bonds. By using the Borel–Cantelli lemma it is easy to see (as in the proof of Proposition 1 of ref. 6) that this implies divergence of the mean size of the cluster of occupied bonds attached to the origin, i.e., Conjecture 1.

The subtlety of the question raised by our stability principle lies in the fact that it is supposed to apply also to clusters whose mean size diverges, yet which are not percolating. One might think that such clusters are only marginally divergent and the smallest additional dilution would render their mean size finite. Our principle says that this will not be the case, except possibly at nongeneric critical points.

To put our conjecture into perspective, let us recall the following rigorously established facts, which all can be interpreted as expressing some kind of stability against dilution:

(a) Georgii,[1] as mentioned in the beginning, proved that on a randomly bond-diluted lattice the Ising model possesses a phase with long-range order (l.r.o.) provided the dilution probability $p < p_c$.

(b) Russo[6] proved that for independent percolation the probability that the origin belongs to the percolating cluster is a smooth function of the bond occupation probability $1 - p$ for all $p < p_c$.

(c) De Masi *et al.*[7] proved that on a lattice on which bonds have been diluted randomly with probability p, the Laplacian retains its continuous spectrum for $p < p_c$.

Let us look at some examples where we have stability of clusters that have divergent mean size, yet do not percolate:

Example 1. Consider the 2D Ising model at inverse temperature $\beta = \beta_c$ and its hemispherical clusters ($+$ and $-$ clusters regarded as bond clusters) as well as their bond dilution with probability p'. The hemispherical clusters do not percolate at β_c.[8] The Fortuin–Kasteleyn clusters[5] are obtained by diluting with $p' = \exp(-2\beta)$ and their mean cluster size is equal to the susceptibility,[5] hence divergent. So Conjecture 1 holds.

Example 2. Consider the 2D Ising model at inverse temperature $\beta > \beta_c$ with $+$ b.c. and the clusters of $+$ spins, as well as their bond dilution with probability p'. It is known that $P(L, \beta, 0) = 1$ for all L.[8] The Fortuin–Kasteleyn clusters[5] are obtained by putting $p' = \exp(-2\beta)$; because there is l.r.o. and the connectivity function is equal to the untruncated 2-point function,[5] the occupied bonds percolate, the unoccupied bonds form islands, and $P(L, \beta, p') = 1$ for $p' \leqslant \exp(-2\beta)$ and all L, i.e., Conjecture 1′ holds.

Example 3. Consider the 2D $O(2)$ model at large β. It has been rigorously established that the susceptibility diverges for sufficiently large β.[9] As shown in refs. 2 and 3, one can associate to this model an Ising model which will also have a divergent susceptibility at large β; this implies the divergence of the mean size of the Fortuin–Kasteleyn clusters, which can be considered as dilutions of the "hemispherical clusters."[2–4] So a variant of Conjecture 1 holds in this example (the dilution is not quite of the form assumed in the conjecture).

We find our principle extremely plausible and believe that arguments similar to Russo's could lead to a rigorous proof. In this paper, however, we present only the results of a numerical investigation of the bond-diluted $+$ or $-$ clusters of the Ising model on the triangular (T) and square (S) lattices. The model is defined as follows: we first consider the $+$ or $-$ clusters regarded as bond clusters (=hemispherical clusters). We then delete bonds independently with probability p'. We thus obtain models depending on two parameters: the inverse temperature β of the Ising model and p'. We sometimes use instead of β the probability $p = \exp(-2\beta)$ to parametrize the models.

The models are described in terms of the following joint probability measure for the spins σ_x (taking values $+1$ or -1) and bond occupation numbers n_{xy} taking values 0 or 1:

$$P(n_{xy}, \sigma_x) = Z^{-1} \prod_{\langle xy \rangle} \{[n_{xy}(1-p') + (1-n_{xy})\, p']\, \delta_{\sigma_x \sigma_y} + (1-n_{xy})\, p(1-\delta_{\sigma_x \sigma_y})\} \tag{1}$$

This measure defines bond clusters in the usual way; we define their size as the number of lattice sites contained in them. Putting $p = p'$, one obtains the probability measure related to the Fortuin–Kasteleyn (FK) representation[5] of the Ising model dubbed Fortuin–Kasteleyn–Swendsen–Wang measure in ref. 10. It is easy to sum over the spins in that case to obtain the well-known FK measure for correlated percolation associated with the Ising model

$$P(n_{xy}) = Z^{-1} 2^{N_c} \prod_{\langle xy \rangle} \{n_{xy}(1-p) + (1-n_{xy})\, p\} \tag{2}$$

where N_c denotes the number of clusters (connected components). On the other hand, putting $p' = 0$ in (1), one obtains the bond percolation model induced by the + and − clusters (H clusters) of the Ising model.

In general we cannot carry out the sum over the spins explicitly, but it is possible to make a general statement based on the FKG inequalities[11]:

Proposition 1. The probability measure given by (1) is increasing in the FKG sense in $1 - p'$.

This implies in particular the intuitively obvious fact that the expected cluster size is increasing with decreasing dilution probability p'. The following conjecture is also intuitively obvious and seen in every numerical experiment (although we do not have a proof):

Conjecture 2. The mean cluster size is increasing with $1 - p$, i.e., with β.

If we denote by $p'_c(p)$ the infimum of the values of p' such that at fixed p the expected size of the cluster of occupied bonds (i.e., bonds $\langle xy \rangle$ with $n_{xy} = 1$) is finite, we can thus make the following general statements:

1. The mean cluster size is monotonically decreasing in p'.
2. $p'_c(p)$ is a monotonically nondecreasing function of p.
3. $p'_c(p) \geqslant p$ at $p = \exp(-2\beta_c)$, where β_c is the critical inverse temperature of the Ising model on the lattice considered.
4. $p'_c(p)(0) = 1 - p_{c,\text{Bernoulli}}$, where the latter is the critical probability for Bernoulli percolation on the lattice considered.

Statement 1 follows from Proposition 1, statement 2 from Conjecture 1, and statement 3 from the fact that at the critical point of the Ising model the mean size of the FK cluster diverges (it is equal to the susceptibility) and there is no percolation of the hemispherical (i.e., + or −) clusters (see Example 1 above).

In the following we will present numerical data that give a rough idea of the shape of the critical curve $p'_c(p)$. In particular, we find that both on the T and S lattices $p'_c(p) > 0$ for p in a nontrivial interval $\exp(-2\beta_c) < p < p_H$, where p_H is the infimum of the values of p such that the expected size of the "hemispherical clusters" (i.e., the + and − clusters) is finite. This implies that for both lattices there is a 2-dimensional "critical region" given by $\exp(-2\beta_c) < p < p_H$ and $p' < p'_c(p)$ in which there is no percolation, but the expected size of the clusters diverges, and confirms our stability principle (Conjecture 1).

2. NUMERICAL RESULTS

2.1. Triangular Lattice

The triangular lattice is self-matching in the terminology of ref. 12. This means that for the site problem there is no difference between ordinary and *-percolation. Therefore, for $p > \exp(-2\beta_c)$ (i.e., $\beta < \beta_c$), by symmetry, there is neither percolation of + sites nor *-percolation of − sites. By the result of Russo already quoted (Proposition 1 in ref. 6), this implies divergence of the expected size of the + and − site clusters which is equivalent to divergence of the expected size of the "hemispherical" (H) clusters of bonds defined above. In other words, on the T lattice, $p_H \leqslant 0$ (presumably it is $=0$).

To determine $p'_c(p)$ for $p > \exp(-2\beta_c)$, i.e., in the high-temperature region, we measured the mean cluster size for of $p' = 1/\sqrt{3}$, 0.5, 0.4, and 0.3 for a sequence of values of β chosen such that the mean cluster size $\langle C \rangle$ remains finite as the size L^2 of the lattice goes to ∞. In practice it turned out that $\langle C \rangle$ reached its infinite-volume limit (within numerical accuracy) once L^2 was 20 times $\langle C \rangle$. To generate the spin configurations, we used Wolff's single cluster variant[13] of the Swendsen–Wang algorithm[14]; since this method relies on the construction of the FK clusters, it is a trivial matter to construct also the clusters determined by the measure given in (1) and measure their size.

The resulting data were fitted to a power law

$$p' = a[\beta - \beta_c(p')]^{-\gamma} \tag{3}$$

[$\beta_c(p')$ is determned by the fit; its meaning is that it determines the value $p_c(p')$ at which the expected cluster size diverges for given p'.]

On the T lattice

$$\beta_c = \tfrac{1}{4} \ln 3 = 0.27465 \tag{4}$$

(see ref. 15, p. 87), so that $\exp(-2\beta_c) = 1/\sqrt{3} = 0.57735$. This is the first value chosen for p' and the purpose is to test the method, because at this p', $\beta_c(p') = \beta_c$. Our data show that we get a quite accurate determination of $\beta_c(p')$. We obtained

$$\beta_c = 0.2722(4), \qquad \gamma = 1.316 \tag{5}$$

The exact value is quite close to the best fit value, but not within the quoted statistical error. The reason is the unavoidable systematic error produced by fitting with a pure power law. For the other values of p' we obtained

$$\beta_c(0.5) = 0.2630(6), \qquad \gamma = 1.398 \tag{6}$$

$$\beta_c(0.4) = 0.2524(14), \qquad \gamma = 1.484 \tag{7}$$

$$\beta_c(0.3) = 0.2402(26), \qquad \gamma = 1.518 \tag{8}$$

Put differently, we have obtained

$$p'_c(0.5910 \pm 0.0007) = 0.5 \tag{6'}$$

$$p'_c(0.6036 \pm 0.0017) = 0.4 \tag{7'}$$

$$p'_c(0.6185 \pm 0.0032) = 0.3 \tag{8'}$$

(The actual measured values of $\langle C \rangle$ are given in Table I.)

It is seen clearly that $\beta_c(p')$ is increasing significantly (far beyond the statistical and systematic error) with p'.

To get a better idea of the shape of the critical curve $p'(p)$, we also measured the mean cluster size for the three values $\beta = 0$, $\beta = \beta_c = 0.2747$, and $\beta = 0.29$, varying p', and fitted the data with a power law

$$\langle C \rangle = b[p' - p'_c(\beta)]^{-\gamma} \tag{9}$$

We obtained

$$p'(0) = -0.028(5), \qquad \gamma = 2.878 \tag{10}$$

$$p'(0.2747) = 0.575(7), \qquad \gamma = 3.024 \tag{11}$$

$$p'(0.29) = 0.610(4), \qquad \gamma = 2.657 \tag{12}$$

(The actual measured values of $\langle C \rangle$ are given in Table II.)

Again we see a systematic error showing up in the slightly negative value obtained for $\beta = 0$. The true value is probably 0, since at $\beta = 0$ we are really studying Bernoulli site percolation with probability 0.5, which is the

Table I. Triangular Lattice, Cluster Size vs. β for $p' = 0.3$, 0.4, 0.5, $1/\sqrt{3} = 0.57735$

$p' = 0.3$					
β	0.	0.05	0.10	0.15	0.20
$\langle C \rangle$	38.61(42)	53.8(1.4)	86.0(1.7)	171.1(2.3)	577.7(13.7)
L	30	40	40	60–80	100
$p' = 0.4$					
β	0.05	0.10	0.15	0.20	0.23
$\langle C \rangle$	23.82(42)	36.50(66)	64.28(63)	189.0(6.1)	615.9(14.0)
L	40	40	60–80	100	100
$p' = 0.5$					
β	0.10	0.15	0.20	0.23	0.25
$\langle C \rangle$	16.49(14)	26.48(36)	59.67(65)	154.9(1.7)	536.8(13.9)
L	20	30	40	60	110
$p' = 0.57735$					
β	0.20	0.23	0.25	0.26	0.265
$\langle C \rangle$	24.59(23)	49.43(28)	114.3(1.9)	261.2(5.1)	501.8(11.1)
L	20	30	40	60	110

Table II. Triangular Lattice: Cluster Size vs. p' for $\beta = 0.$, 0.27465, 0.29

$\beta = 0$					
p'	0.40	0.30	0.20	0.15	0.10
$\langle C \rangle$	18.25(21)	38.92(34)	109.95(56)	220.0(2.3)	568.6(2.6)
L	30	30	48–50	80	110
$\beta = 0.27465$					
p'		0.70	0.67		0.64
$\langle C \rangle$		25.61(21)	58.72(41)		185.0(2.6)
L		20–40	40–60		60
$\beta = 0.29$					
p'	0.73	0.70	0.67		0.65
$\langle C \rangle$	18.17(40)	37.99(35)	115.9(2.6)		328.8(4.7)
L	20	30	60		60

critical value. However, we are diluting bonds, not sites; we do not know for sure if the clusters are marginally stable against the former process, hence the true value could be slightly positive rather than 0. In any case, the results (6′)–(8′) and (10)–(12) together give some general picture of the shape of the critical curve $p'(p)$ that is in agreement with our stability principle.

It turns out that the exponent γ is varying rapidly near the diagonal $p = p'$ (in addition to the values obtained from fitting "vertically," i.e., at fixed p, and "horizontally," i.e., at fixed p', we know the exact value of the exponent along the diagonal, since it is equal to the exponent of the susceptibility, 1.75). This rapid change in γ is accompanied by a fast change of the slope of $p'(p)$. We do not know if there is in fact a singularity of the critical curve on the diagonal.

This slope of the critical curve is the reason for our using "vertical" fits in one regime and "horizontal" ones elsewhere, since the true power behavior will be obscured if one is approaching the critical curve in a nearly tangential direction.

2.2. Square Lattice

The square lattice is not self-matching and at small β the mean size of the H ($+$ or $-$) clusters is finite, whereas there is *-percolation of both the $+$ and $-$ *-clusters.

A priori we do not know if there is a value $\beta_H < \beta_c$ such that the mean size of the H clusters first diverges for $\beta > \beta_H$ and not for $\beta < \beta_H$. If one believes in a strong form of universality, however, the qualitative behavior of the phase diagram should be as on the T lattice, which would mean that there is an interval below β_c showing divergent mean size of the H ($+$ or $-$) clusters. Our stability principle then requires that $p_c'(\beta) > 0$ for β in the interval $(\beta_H, \beta_c]$.

To find out if in fact $\beta_H < \beta_c$, we determined the size of the H clusters at $\beta = 0$, 0.1, 0.2, 0.225, 0.25, 0.26 and fitted the data to a power law singularity. We obtained $\beta_H = 0.42(2)$ with an exponent $\gamma = 3.144$. Since $\beta_c = 0.4407$, this means that by this method we are unable to decide the question without spending a lot more computer time. The measured values of the mean cluster size appear in Table III.

Table III. Square Lattice: Cluster Size vs. β for $p' = 0$ (H clusters)

β	0.	0.10	0.20	0.225	0.25	0.26
$\langle C \rangle$	58.4(1.0)	136.3(3.7)	443(14)	690(29)	941(35)	1239(37)
L	62	62	126	190	254	260

Table IV. Square Lattice: Cluster Size vs. p' for $p' = 0.3$, 0.41421

$p' = 0.3$					
β	0.25	0.30	0.35	0.38	0.39
$\langle C \rangle$	33.95(58)	59.71(97)	143.4(3.1)	335.2(4.7)	511.2(13.0)
L	30	60	60	100	110
$p' = 0.41421$					
β	0.30	0.35	0.40	0.42	0.425
$\langle C \rangle$	20.62(20)	37.85(50)	114.4(2.3)	315.0(4.7)	482.5(6.7)
L	60	60	60	100	110

So we proceeded as on the T lattice, measuring $\langle C \rangle$ for varying β ("horizontal fit") at $p' = \exp(-2\beta_c) = 0.4142$ and $p' = 0.3$ (the first value again to test the reliability of the fits).

The result is

$$\beta_c(0.4142) = 0.438(1), \qquad \gamma = 1.335 \tag{13}$$

$$\beta_c(0.3) = 0.426(4), \qquad \gamma = 1.709 \tag{14}$$

The measured values of the mean cluster size are given in Table IV. As before, there is a systematic deviation from the exact value $\beta_c = 0.4407$ in the fit (13), but the difference between the two values of β_c is much larger and suggests strongly that $p_c'(\beta) > 0$ for some $\beta < \beta_c$. By the general monotonicity argument mentioned in the beginning, this implies $\beta_H < \beta_c$. So the data also confirm the general stability principle for the S lattice.

3. CONCLUSIONS

Our findings mean, among other things, that the phase diagram in the (p, p') plane is qualitatively the same, independent of the lattice structure. Additional support for this kind of universality comes from the following: we measured the size of the H and FK clusters, denoted $\langle \mathrm{H} \rangle$ and $\langle \mathrm{FK} \rangle$, on both lattices right at the critical point and found that not only

$$\langle \mathrm{H} \rangle(L) = cL^{2-\eta_H} \tag{15}$$

as well as

$$\langle \mathrm{FK} \rangle(L) = cL^{2-\eta} \tag{16}$$

with nearly the same values of η (about 0.25) and η_H (about 0.1) on both lattices, but that also the absolute numbers agree within numerical accuracy for the two lattices.

Table V. Comparison of the FK and H Clusters for the T and S Lattices at $\beta = \beta_c$ for Lattices of size $L = 10, 20, 40, 80$

T lattice				
L	10	20	40	80
$\langle H \rangle$	78.98(16)	290.0(1.6)	1079.7(6.7)	4097(22)
$\langle FK \rangle$	59.71(37)	199.6(2.3)	677.4(8.5)	2284(32)
S lattice				
L	10	20	40	80
$\langle H \rangle$	80.09(37)	296.2(1.5)	1093.0(9.8)	4023(34)
$\langle FK \rangle$	62.07(37)	203.7(2.8)	677.7(10.2)	2283(64)

We give the actual measured numbers in Table V.

To sum up: we have found on both the T and S lattices numerical evidence for the existence of a two-dimensional region in the (p, p') plane in which the "diluted H clusters" have divergent mean size, yet do not percolate. In particular, this means that on the S lattice, contrary to some expectations, there is a $\beta_H < \beta_c$ such that $\langle H \rangle$ diverges for $\beta > \beta_H$.

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